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THE CAYLEYAN CURVE OF THE QUARTIC

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The quartic curve $(\alpha x)^4 = 0$ determines a correspondence

$$(\alpha x)^2(\alpha y)\alpha_i = 0, \qquad i = 0, 1, 2,$$

in which x is a point on the Hessian curve and y a point of the Steinerian. The locus of lines ξ joining corresponding points x, y is the Cayleyan, known to be of degree 18 in ξ and 12 in the coefficients of the quartic.

The Cayleyan can be expressed in terms of the two contravariants of the quartic, $(s\xi)^4\ [\equiv \frac{1}{2}|\alpha\beta\xi|^4]$ and $(t\xi)^6\ [\equiv \frac{1}{6}|\beta\gamma\xi|^2|\gamma\alpha\xi|^2|\alpha\beta\xi|^2]$, and of terms produced by operating with the polars of these on $(\alpha x)^4$. The working out of this depends on a special reference triangle which is always valid for the general quartic. Suppose

$$\begin{aligned}(ax)^4 \equiv\ & ax_0^4 + 4a_1x_0^3x_1 + 4a_2x_0^3x_2 + 6hx_0^2x_1^2 + 12lx_0^2x_1x_2 + 6gx_0^2x_2^2 \\ & + 4b_0x_0x_1^3 + 12mx_0x_1^2x_2 + 12nx_0x_1x_2^2 + 4c_0x_0x_2^3 \\ & + bx_1^4 + 4b_2x_1^3x_2 + 6fx_1^2x_2^2 + 4c_1x_1x_2^3 + cx_2^4.\end{aligned}$$

Since x and y as given above can never coincide for the general quartic, because

$$(\alpha x)^3\alpha_i = 0, \qquad i = 0, 1, 2,$$

is the condition that $(\alpha x)^4$ have a double point, let x, the point of the Hessian, be the reference point $(0, 1, 0)$ and let y, the point of the Steinerian, be $(0, 0, 1)$, so that $x_0 = 0$ is a line of the Cayleyan. Then

$$\begin{aligned}\alpha_1^2\alpha_2\alpha_i &= 0, \qquad i = 0, 1, 2, \\ \text{or} \quad m &= b_2 = f = 0.\end{aligned}$$

This reference scheme is maintained throughout, though more highly specialized as occasion demands. Under it the Hessian becomes

$2n(bh - b_0^2)x_1^5x_0 + 2c_1(bh - b_0^2)x_1^5x_2 +$ lower powers of x_1;
$(s\xi)^4 \equiv bc\xi_0^4 - 4cb_0\xi_0^3\xi_1 + 4(-bc_0 + b_0c_1)\xi_0^3\xi_2 + \ldots\ldots;$
$(t\xi)^6 \equiv -bc_1^2\xi_0^6 + 2(-bcn + bc_0c_1 + 2b_0c_1^2)\xi_0^5\xi_1 + 6bc_1n\xi_0^5\xi_2 + \ldots\ldots$

In the first place, the Cayleyan is known to be on the stationary lines of the quartic, which are the common lines of $(s\xi)^4$ and $(t\xi)^6$. Therefore the Cayleyan must be made up of terms containing either $(s\xi)^4$ or $(t\xi)^6$ at least once.

Now let us see what are the common lines of the Cayleyan and $(s\xi)^4$. To make x_0 a line of the latter requires that $bc = 0$. If $b = 0$, then x_0 is a stationary line of the quartic. If $c = 0$, then not only is x_0 a line of $(s\xi)^4$, but its contact with it is $(0, 0, 1)$, the point of the Steinerian, which has also become a point of the quartic. Therefore quartic, Steinerian, and $(s\xi)^4$ all meet in a point. The Steinerian, a curve of order 12, meets the quartic in 48 points; the 48 corresponding lines together with the 24 stationary lines make up the 72 common lines of the Cayleyan and $(s\xi)^4$. The condition that the polar point of $(s\xi)^4$ lie on $(\alpha x)^4$ is the vanishing of $(s\xi)^3(s\alpha)\ (s'\xi)^3(s'\alpha)(s''\xi)^3(s''\alpha)(s'''\xi)^3(s'''\alpha)$; this, when multiplied by $(t\xi)^6$, is of proper degree for a term of the Cayleyan. It is, then, the only term of the Cayleyan not containing $(s\xi)^4$.

It is now in order to ask for the common lines of the Cayleyan and $(t\xi)^6$. For x_0 to be a line of the latter requires that $bc_1^2 = 0$. Again setting aside the stationary lines, we have $c_1 = 0$. Then it is seen that x_0 has as its contact with $(t\xi)^6$ the point $(0, 1, 0)$; furthermore, it is tangent to the Hessian at the same point. Therefore there are a certain number of lines of the Cayleyan which are also lines of both the Hessian and $(t\xi)^6$, these two curves having contact on these lines. For the terms of the Cayleyan not containing $(t\xi)^6$ it is sufficient to use

$$\begin{aligned}
&\lambda(s\xi)^4.(s'\xi)^3(s'\alpha)(s''\xi)^3(s''\alpha)(s'''\xi)^3(s'''\alpha)(t\xi)^5(t\alpha)\\
&+ (s\xi)^4(s'\xi)^4\,[\sigma(s''\xi)^3(s''\alpha)(s'''\xi)^3(s'''\alpha)(t\xi)^4(t\alpha)^2\\
&\quad + \tau(s''\xi)^3(s''\alpha)(s'''\xi)^2(s'''\alpha)^2(t\xi)^5(t\alpha)]\\
&+ (s\xi)^4(s'\xi)^4(s''\xi)^4\,[\varphi(s'''\xi)^2(s'''\alpha)^2(t\xi)^4(t\alpha)^2 + \mu(s'''\xi)^3(s'''\alpha)(t\xi)^3(t\alpha)^3]\\
&+\nu\,(s\xi)^4(s'\xi)^4(s''\xi)^4(s'''\xi)^4.(t\xi)^2(t\alpha)^4,
\end{aligned}$$

where λ, σ, τ, φ, μ, ν are undetermined coefficients. By requiring the highest power of ξ_0 to vanish when $c_1 = 0$ certain relations on these coefficients are obtained, not enough to solve, however. To the terms given above it is necessary to add terms containing $(t\xi)^6$;

$$\begin{aligned}
&\rho(t\xi)^6\ .\ (s\xi)^3(s\alpha)(s'\xi)^3(s'\alpha)(s''\xi)^3(s''\alpha)(s'''\xi)^3(s'''\alpha)\\
&+ \epsilon(s\xi)^4(s'\xi)^4(s''\xi)^4(t\xi)^6\ .\ (s'''\alpha)^4
\end{aligned}$$

will be found sufficient.

There are certain other lines known to be lines of the Cayleyan. There are 21 points whose polar cubics as to the quartic break up into a conic and a line, which is a fourfold line of the Cayleyan. Let one of the 21 points be $(1, 0, 0)$; then for the polar cubic to contain x_0 as a factor requires that $b_0 = c_0 = n = 0$. Now, using this condition, require that the highest power of ξ_0 in the expression for the Cayleyan vanish. The result will be certain conditions on the undetermined coefficients, but still not enough to solve.

Instead of putting the Cayleyan again on these lines it is easier to proceed at once to the general reference scheme which has been the basis of all the work, when only $m = b_2 = f = 0$, and require that the highest coefficient of ξ_0 vanish. This at once completes the work and furnishes proof of its correctness. The Cayleyan is obtained as

$$\begin{aligned}
&33\,(s\xi)^4 \,.\, (s'\xi)^3(s'\alpha)(s''\xi)^3(s''\alpha)(s'''\xi)^3(s'''\alpha)(t\xi)^5(t\alpha) \\
&- \tfrac{22}{3}\,(t\xi)^6 \,.\, (s\xi)^3(s\alpha)(s'\xi)^3(s'\alpha)(s''\xi)^3(s''\alpha)(s'''\xi)^3(s'''\alpha) \\
&+ (s\xi)^4(s'\xi)^4\,[15(s''\xi)^3(s''\alpha)(s'''\xi)^3(s'''\alpha)(t\xi)^4(t\xi)^2 \\
&\quad - 57(s''\xi)^3(s''\alpha)(s'''\xi)^2(s'''\alpha)^2(t\xi)^5(t\alpha)\,] \\
&+ (s\xi)^4(s'\xi)^4(s''\xi)^4\,[\tfrac{35}{2}\,(s'''\xi)^2(s'''\alpha)^2(t\xi)^4(t\alpha)^2 - 10(s'''\xi)^3(s'''\alpha)(t\xi)^4(t\alpha)^3] \\
&+ \tfrac{65}{18}\,(s\xi)^4(s'\xi)^4(s''\xi)^4(t\xi)^6 \,.\, (s\alpha)^4 \\
&- \tfrac{5}{3}\,(s\xi)^4(s'\xi)^4(s''\xi)^4(s'''\xi)^4 \,.\, (t\xi)^2(t\alpha)^4.
\end{aligned}$$

Since, however, this expression has been obtained by causing a coefficient to vanish, there is the possibility that it gives merely a syzygy and vanishes identically. Therefore it was tested on the special quartic $x_0^4 + x_1^4 + x_2^4$, where the Cayleyan is known to be $\xi_0^2\xi_1^2\xi_2^2$, and found not to vanish.

The stationary lines of the quartic are known to be lines of the Steinerian. From the above form of the Cayleyan it can be shown that the contacts of these lines are the same for the Cayleyan as for the Steinerian, so that the two curves touch. We had also certain common lines of the Cayleyan and Hessian, which were likewise lines of $(t\xi)^6$. These lines can be shown to have the same contact as to the three curves. The Cayleyan and $(t\xi)^6$ have 108 common lines, 24 of which are absorbed by the flexes, leaving 84 to be accounted for here. Because of the contact of the curves each line counts for two common lines; therefore the Cayleyan, Hessian, and $(t\xi)^6$ touch in 42 points.

Certain interesting facts come up under the reference scheme here employed. The polar conic of $(0, 1, 0)$ is

$$hx_0^2 + 2b_0x_0x_1 + bx_1^2 = 0,$$

two lines, which coincide if

$$bh - b_0^2 = 0.$$

But this requires the Hessian to have a double point; therefore the general quartic cannot have a polar conic made up of two coincident lines. Also the polar cubic of (0, 0, 1) is

$$a_2x_0^3 + 3lx_0^2x_1 + 3gx_0^2x_2 + 6nx_0x_1x_2 + 3c_0x_0x_2^2 + 3c_1x_2^2 + cx_2^3 = 0.$$

This can have a cusp only if

$$c_1l - n^2 = 0.$$

This has clearly nothing to do with $b = 0$, the condition that x_0 be a stationary line of the quartic. Therefore the cusps of the Steinerian do not lie on the stationary lines, as might be expected from their number —twenty-four. $n = 0$ is the condition that $x_2 = 0$ be the tangent to the Hessian; then the cusp cannot be obtained by making $c_1 = 0$, for then the Hessian has a double point. Putting $l = 0$ shows that the cusp tangent is also the tangent to the Hessian. Use of $n = 0$ also shows that the polar points of lines of the Cayleyan as to $(t\xi)^6$ lie on the corresponding tangents to the Hessian.

A SEARCH FOR AN EINSTEIN RELATIVITY-GRAVITATIONAL EFFECT IN THE SUN

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From the equivalence principle of generalized relativity Einstein[1] concludes that the propagation of light is influenced by gravitation, and deduces two important consequences that can be subjected to the test of observation; namely, a train of light waves passing close to the edge of the sun is refracted so that the angular distance of a star appearing near the sun is increased by 1″.75, and the Fraunhofer lines are displaced to the red in the solar spectrum by an amount equivalent to a velocity of recession of 0.634 km/sec. The amount depends only on the difference in gravitational potential between the gravitation field in which the radiation originates and the field where it is received. In the case of massive stars with density comparable to that of the sun the line displacement may be large, equivalent to 0.634 km/sec. $M^{2/3}\, d^{1/3}$, where M and d are in terms of the sun's mass and density.[2] Confirmation of either of these consequences would have not only an important bearing upon the establishment of the relativity principle but also upon the in-